

Radial Limits of Capillary Surfaces at Corners

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Abstract

Consider a solution $f \in C^2(\Omega)$ of a prescribed mean curvature equation

$$\operatorname{div} \left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = 2H(x, f) \quad \text{in } \Omega \subset \mathbb{R}^2,$$

where Ω is a domain whose boundary has a corner at $\mathcal{O} = (0, 0) \in \partial\Omega$ and the angular measure of this corner is 2α , for some $\alpha \in (0, \pi)$. Suppose $\sup_{x \in \Omega} |f(x)|$ and $\sup_{x \in \Omega} |H(x, f(x))|$ are both finite. If $\alpha > \frac{\pi}{2}$, then the (nontangential) radial limits of f at \mathcal{O} ,

$$Rf(\theta) \stackrel{\text{def}}{=} \lim_{r \downarrow 0} f(r \cos(\theta), r \sin(\theta)),$$

were recently proven by the authors to exist, independent of the boundary behavior of f on $\partial\Omega$, and to have a specific type of behavior.

Suppose $\alpha \in (\frac{\pi}{4}, \frac{\pi}{2}]$, the contact angle $\gamma(\cdot)$ that the graph of f makes with one side of $\partial\Omega$ has a limit (denoted γ_2) at \mathcal{O} and

$$\pi - 2\alpha < \gamma_2 < 2\alpha.$$

We prove that the (nontangential) radial limits of f at \mathcal{O} exist and the radial limits have a specific type of behavior, independent of the boundary behavior of f on the other side of $\partial\Omega$. We also discuss the case $\alpha \in (0, \frac{\pi}{2}]$.

1 Introduction and Statement of Main Theorems

Let Ω be a domain in \mathbb{R}^2 whose boundary has a corner at $\mathcal{O} \in \partial\Omega$. Suppose $H : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and H satisfies one of the conditions which guarantees that “cusp solutions” (e.g. §5 of [7], [9]) do not exist; for example, $H(\mathbf{x}, t)$ is strictly increasing in t for each \mathbf{x} or is real-analytic (e.g. constant). We will assume $\mathcal{O} = (0, 0)$. Let $\Omega^* = \Omega \cap B_{\delta^*}(\mathcal{O})$, where $B_{\delta^*}(\mathcal{O})$ is the ball in \mathbb{R}^2 of radius δ^* about

\mathcal{O} . Polar coordinates relative to \mathcal{O} will be denoted by r and θ . We assume that $\partial\Omega$ is piecewise smooth and there exists $\alpha \in (0, \pi)$ such that $\partial\Omega \setminus \{\mathcal{O}\} \cap B_{\delta^*}(\mathcal{O})$ consists of two (open) C^1 arcs $\partial^+\Omega^*$ and $\partial^-\Omega^*$, whose tangent lines approach the lines $L^+ : \theta = \alpha$ and $L^- : \theta = -\alpha$, respectively, as the point \mathcal{O} is approached.

Suppose $\alpha > \frac{\pi}{2}$, $f \in C^2(\Omega)$ satisfies the prescribed mean curvature equation

$$Nf(x) = 2H(x, f(x)) \quad \text{for } x \in \Omega, \quad (1)$$

where $Nf = \nabla \cdot Tf = \operatorname{div}(Tf)$ and $Tf = \frac{\nabla f}{\sqrt{1+|\nabla f|^2}}$, and

$$\sup_{x \in \Omega} |f(x)| < \infty \quad \text{and} \quad \sup_{x \in \Omega} |H(x, f(x))| < \infty. \quad (2)$$

In [4], the authors proved that the radial limits

$$Rf(\theta) \stackrel{\text{def}}{=} \lim_{r \downarrow 0} f(r \cos(\theta), r \sin(\theta))$$

exist for all $\theta \in (-\alpha, \alpha)$, $Rf(\cdot)$ is a continuous function on $(-\alpha, \alpha)$ and these radial limits have similar behavior to that observed in Theorem 1 of [7].

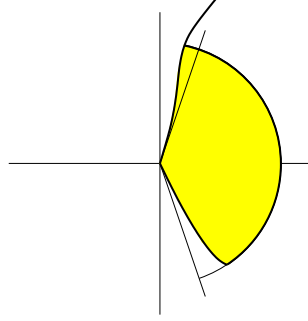


Figure 1: The domain Ω^*

Suppose $\alpha \leq \frac{\pi}{2}$ (see Figure 1) and $f \in C^2(\Omega) \cap C^1(\Omega \cup \partial^-\Omega^*)$ satisfies (1) and (2). In [4], it is shown that if

$$\lim_{\partial^-\Omega^* \ni \mathbf{x} \rightarrow \mathcal{O}} f(\mathbf{x}) = z_2 \quad \text{exists,} \quad (3)$$

then the radial limits of f at \mathcal{O} exist and behave as expected. In this paper, we consider the capillary problem as our model and suppose $f \in C^2(\Omega) \cap C^1(\Omega \cup \partial^-\Omega^*)$ satisfies (1), (2) and the boundary condition

$$Tf(x) \cdot \nu(x) = \cos(\gamma(x)) \quad \text{for } x \in \partial^-\Omega^*, \quad (4)$$

where $\nu(x)$ is the exterior unit normal to Ω at $x \in \partial\Omega$ and $\gamma : \partial\Omega \rightarrow [0, \pi]$ is the contact angle between the graph of f and $\partial\Omega \times \mathbb{R}$, and

$$\lim_{\partial^-\Omega^* \ni \mathbf{x} \rightarrow \mathcal{O}} \gamma(\mathbf{x}) = \gamma_2. \quad (5)$$

We shall prove

Theorem 1. Let $f \in C^2(\Omega) \cap C^1(\Omega \cup \partial^-\Omega^*)$ satisfy (1) & (4) and suppose (2) and (5) hold, $\alpha \in (\frac{\pi}{4}, \frac{\pi}{2}]$ and

$$\pi - 2\alpha < \gamma_2 < 2\alpha. \quad (6)$$

Then (3) holds, $Rf(\theta)$ exists for all $\theta \in (-\alpha, \alpha)$ and $Rf(\cdot)$ is a continuous function on $[-\alpha, \alpha)$, where $Rf(-\alpha) \stackrel{\text{def}}{=} z_2$. Further $Rf(\cdot)$ behaves in one of the following ways:

- (i) $Rf : [-\alpha, \alpha) \rightarrow \mathbb{R}$ is a constant function (hence f has a nontangential limit at \mathcal{O}).
- (ii) There exist α_1 and α_2 so that $-\alpha \leq \alpha_1 < \alpha_2 \leq \alpha$ and Rf is constant on $[-\alpha, \alpha_1]$ and $[\alpha_2, \alpha)$ and strictly increasing or strictly decreasing on $[\alpha_1, \alpha_2]$.

If $\alpha \in (0, \frac{\pi}{4}]$, then (6) cannot be satisfied. If $\alpha \in (\frac{\pi}{4}, \frac{\pi}{2}]$ but $\gamma_2 \geq 2\alpha$ or $\gamma_2 \leq \pi - 2\alpha$, then (6) is not satisfied. In both cases, Theorem 1 is not applicable. In these cases, we can prove the existence of $Rf(\cdot)$ if we add an assumption about the behavior of γ on $\partial^+\Omega^*$.

Theorem 2. Let $f \in C^2(\Omega) \cap C^1(\Omega \cup \partial^-\Omega^* \cup \partial^+\Omega^*)$ satisfy (1)-(4). Suppose (2) and (5) hold, $\alpha \in (0, \frac{\pi}{2}]$, there exist $\lambda_1, \lambda_2 \in [0, \pi]$ with $0 < \lambda_2 - \lambda_1 < 4\alpha$ such that $\lambda_1 \leq \gamma(x) \leq \lambda_2$ for $x \in \partial^+\Omega^*$ and

$$\pi - 2\alpha - \lambda_1 < \gamma_2 < \pi + 2\alpha - \lambda_2. \quad (7)$$

Then the conclusions of Theorem 1 hold.

Remark: Theorem 2 only offers a new result when $\lambda_1 = 0$ or $\lambda_2 = \pi$; Figure 8 of [10] illustrates one example in which $\lambda_1 = 0$ or $\lambda_2 = \pi$ occurs. If $0 < \lambda_1 < \lambda_2 < \pi$, then Theorem 2 is a consequence of Theorem 1 of [7]; in this case, the argument here or in [7] implies $Rf(\theta)$ exists for all $\theta \in [-\alpha, \alpha]$.

Remark: In [1, 6], Concus and Finn proved that, in a neighborhood \mathcal{U} of \mathcal{O} and assuming $\partial^+\Omega^*$ and $\partial^-\Omega^*$ are straight line segments, a solution of a constant mean curvature equation (i.e. H is constant in (1)) with constant contact angles γ_1 on $\mathcal{U} \cap \partial^+\Omega^*$ and γ_2 on $\mathcal{U} \cap \partial^-\Omega^*$ can exist only if $|\pi - \gamma_1 - \gamma_2| \leq 2\alpha$. Using this, when $\gamma_1 = 0$, we would obtain a (local) upper bound for f in Theorem 1 when $\pi - 2\alpha < \gamma_2$ and, when $\gamma_1 = \pi$, a (local) lower bound for f when $\gamma_2 < 2\alpha$; these two inequalities are equivalent to (6).

Remark: As in [7], conclusion (3) of Theorems 1 and 2 is a consequence of a general argument; establishing (3) is not a key step in the proof.

Remark: One might contemplate replacing hypothesis (5) by something like $0 < \sigma_1 \leq \gamma(x) \leq \sigma_2 < \pi$ for $x \in \partial^+\Omega^*$ (as in [7]) and suitably modifying (6) or (7). The comparison methods used here allow for this possibility.

2 Preliminary Remarks

Let $f \in C^2(\Omega)$ satisfy (1) and suppose (2) holds. Throughout the remainder of the article, let us assume that $M_1 \in (0, \infty)$, $M_2 \in [0, \infty)$,

$$\sup_{x \in \Omega} |f(x)| \leq M_1 \quad \text{and} \quad \sup_{x \in \Omega} |H(x, f(x))| \leq M_2. \quad (8)$$

2.1 A Specific Torus

We will use portions of tori and comparison function arguments as, for instance, in Examples 2 & 3 of [7] and the Courant-Lebesgue lemma ([2], Lemma 3.1) to obtain upper and lower bounds on f near \mathcal{O} in specific subsets of Ω and prove Theorems 1 and 2. Let us discuss the construction of a particular torus.

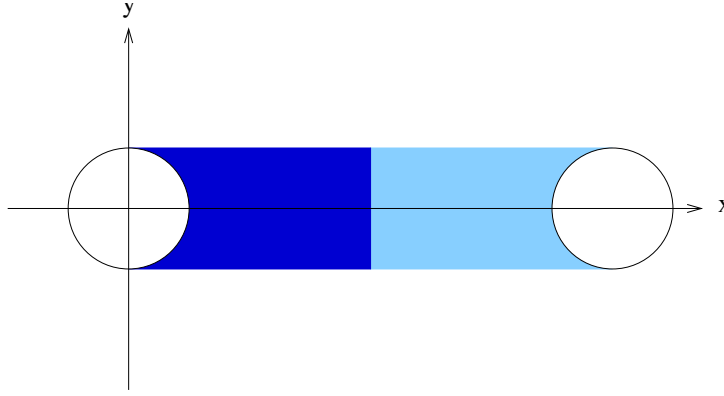


Figure 2: The regions Δ (in dark blue) and Δ^R (in light blue)

Set

$$r_0 = \begin{cases} 1 & \text{if } M_2 = 0 \\ \frac{1}{M_2} + 1 - \sqrt{\left(\frac{1}{M_2}\right)^2 + 1} & \text{if } M_2 > 0. \end{cases}$$

Let $\Delta = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |\mathbf{x}| \geq r_0, 0 \leq x_1 \leq 2, |x_2| \leq r_0\}$ and $\Delta^R = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : (4 - x_1, x_2) \in \Delta\}$. Let

$$\mathcal{T} = \{(2 + (2 + r_0 \cos(v)) \cos(u), r_0 \sin(v), (2 + r_0 \cos(v)) \sin(u)) : u, v \in [0, 2\pi]\}$$

be a torus with axis of symmetry $\{(2, y, 0) : y \in \mathbb{R}\}$, major radius $R_0 = 2$ and minor radius r_0 ; recall that the mean curvature of \mathcal{T} (with respect to the exterior normal) at $(2 + (2 + r_0 \cos(v)) \cos(u), r_0 \sin(v), (2 + r_0 \cos(v)) \sin(u))$ is

$$\frac{1}{2} H_T = -\frac{2 + 2r_0 \cos(v)}{2r_0(2 + r_0 \cos(v))}.$$

A calculation shows that

$$-\left(\frac{1}{r_0} + \frac{1}{2+r_0}\right) \leq H_T \leq -\left(\frac{1}{r_0} - \frac{1}{2-r_0}\right) = -M_2. \quad (9)$$

Set

$$\mathcal{T}^+ = \{(\mathbf{x}, z) \in \mathcal{T} : \mathbf{x} \in \Delta, z \geq 0\} \quad \text{and} \quad \mathcal{T}^- = \{(\mathbf{x}, z) \in \mathcal{T} : \mathbf{x} \in \Delta, z \leq 0\}.$$

Let $h^+, h^- : \Delta \rightarrow \mathbb{R}$ be functions whose graphs satisfy

$$\{(\mathbf{x}, h^+(\mathbf{x})) : \mathbf{x} \in \Delta\} = \mathcal{T}^+ \quad \text{and} \quad \{(\mathbf{x}, h^-(\mathbf{x})) : \mathbf{x} \in \Delta\} = \mathcal{T}^-.$$

Then, from (9), we have

$$\operatorname{div} \left(\frac{h^+}{\sqrt{1+|\nabla h^+|^2}} \right) \geq M_2 \quad \text{and} \quad \operatorname{div} \left(\frac{h^-}{\sqrt{1+|\nabla h^-|^2}} \right) \leq -M_2. \quad (10)$$

For each $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ let $\Delta_\beta = \mathcal{R}_\alpha \circ T_\beta(\Delta)$, where $\mathcal{R}_\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, given by

$$(x_1, x_2) \mapsto (\cos(\alpha)x_1 + \sin(\alpha)x_2, -\sin(\alpha)x_1 + \cos(\alpha)x_2),$$

is the rotation about $(0,0)$ through the angle $-\alpha$ and $T_\beta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, given by

$$(x_1, x_2) \mapsto (x_1 - r_0 \cos(\beta), x_2 - r_0 \sin(\beta)),$$

is the translation taking $(r_0 \cos(\beta), r_0 \sin(\beta)) \in \partial\Delta$ to $(0,0)$. We will let τ_1 denote the angle that upward tangent ray to $T_\beta(C)$ makes with the negative x_1 -axis and let τ_2 denote the angle that upward tangent ray to $T_{-\beta}(C)$ makes with the positive x_1 -axis, where $C = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |\mathbf{x}| = r_0, x_1 \geq 0\}$. (Figure 3 illustrates this when $\beta > 0$.) Let $h_\beta^\pm : \Delta_\beta \rightarrow \mathbb{R}$ be defined by $h_\beta^\pm = h^\pm \circ T_\beta^{-1} \circ \mathcal{R}_\alpha^{-1}$.

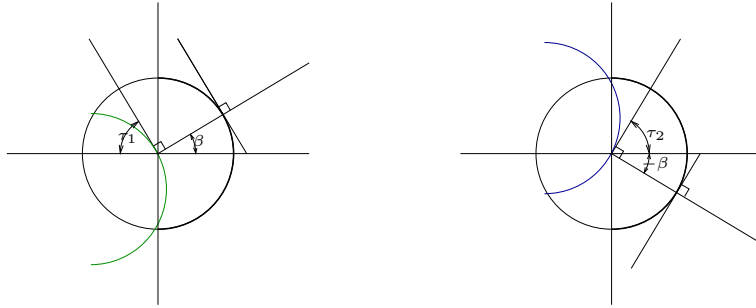


Figure 3: Left: $\beta + \tau_1 = \frac{\pi}{2}$ Right: $-\beta + \tau_2 = \frac{\pi}{2}$ ($\beta \geq 0$)

Let q denote the modulus of continuity of h^- (i.e. $|h_\beta^-(\mathbf{x}_1) - h_\beta^-(\mathbf{x}_2)| \leq q(|\mathbf{x}_1 - \mathbf{x}_2|)$). Notice that q is also the modulus of continuity of h^+ , as well as for h_β^- and h_β^+ for each $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

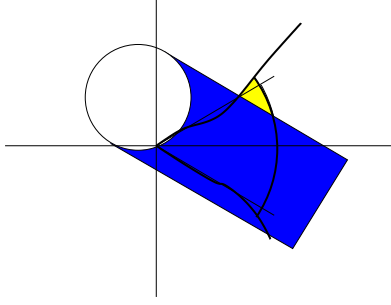


Figure 4: The domain (in blue) of a toroidal function h_{β}^{\pm} , $\alpha < \frac{\pi}{4}$.

2.2 Parametric Framework

Since $f \in C^0(\Omega)$, we may assume that f is uniformly continuous on $\{\mathbf{x} \in \Omega^* : |\mathbf{x}| > \delta\}$ for each $\delta \in (0, \delta^*)$; if this is not true, we may replace Ω with U , $U \subset \Omega$, such that $\partial\Omega \cap \partial U = \{\mathcal{O}\}$ and $\partial U \cap B_{\delta^*}(\mathcal{O})$ consists of two arcs $\partial^+ U$ and $\partial^- U$, whose tangent lines approach the lines $L^+ : \theta = \alpha$ and $L^- : \theta = -\alpha$, respectively, as the point \mathcal{O} is approached. Set

$$S_0^* = \{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in \Omega^*\}$$

and

$$\Gamma_0^* = \{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in \partial\Omega^* \setminus \{\mathcal{O}\}\};$$

the points where $\partial B_{\delta^*}(\mathcal{O})$ intersect $\partial\Omega$ are labeled $A \in \partial^- \Omega^*$ and $B \in \partial^+ \Omega^*$. From the calculation on page 170 of [7], we see that the area of S_0^* is finite; let M_0 denote this area. For $\delta \in (0, 1)$, set

$$p(\delta) = \sqrt{\frac{8\pi M_0}{\ln(\frac{1}{\delta})}}.$$

Let $E = \{(u, v) : u^2 + v^2 < 1\}$. As in [3, 7], there is a parametric description of the surface S_0^* ,

$$Y(u, v) = (a(u, v), b(u, v), c(u, v)) \in C^2(E : \mathbb{R}^3), \quad (11)$$

which has the following properties:

- (a₁) Y is a diffeomorphism of E onto S_0^* .
- (a₂) Set $G(u, v) = (a(u, v), b(u, v))$, $(u, v) \in E$. Then $G \in C^0(\overline{E} : \mathbb{R}^2)$.
- (a₃) Let $\sigma = G^{-1}(\partial\Omega^* \setminus \{\mathcal{O}\})$; then σ is a connected arc of ∂E and Y maps σ strictly monotonically onto Γ_0^* . We may assume the endpoints of σ are \mathbf{o}_1 and \mathbf{o}_2 and there exist points $\mathbf{a}, \mathbf{b} \in \sigma$ such that $G(\mathbf{a}) = A$, $G(\mathbf{b}) = B$, G maps the (open) arc $\mathbf{o}_1 \mathbf{b}$ onto $\partial^+ \Omega$, and G maps the (open) arc $\mathbf{o}_2 \mathbf{a}$ onto $\partial^- \Omega$. (Note that \mathbf{o}_1 and \mathbf{o}_2 are not assumed to be distinct at this point; one of Figure 4a or 4b of [8] illustrates this situation.)

- (a₄) Y is conformal on E : $Y_u \cdot Y_v = 0, Y_u \cdot Y_u = Y_v \cdot Y_v$ on E .
(a₅) $\triangle Y := Y_{uu} + Y_{vv} = H(Y) Y_u \times Y_v$ on E .

Here by the (open) arcs $\mathbf{o}_1 \mathbf{b}$ and $\mathbf{o}_2 \mathbf{a}$ are meant the component of $\partial E \setminus \{\mathbf{o}_1, \mathbf{b}\}$ which does not contain \mathbf{a} and the component of $\partial E \setminus \{\mathbf{o}_2, \mathbf{a}\}$ which does not contain \mathbf{b} respectively. Let $\sigma_0 = \partial E \setminus \sigma$.

There are two cases we will need to consider during the proofs of Theorem 1 and Theorem 2:

(A) $\mathbf{o}_1 = \mathbf{o}_2$.

(B) $\mathbf{o}_1 \neq \mathbf{o}_2$.

These correspond to Cases 5 and 3 respectively in Step 1 of the proof of Theorem 1 of [7].

3 Proof of Theorem 1

Since $\pi - 2\alpha < \gamma_2 < 2\alpha$, we can choose $\tau_1 \in (\pi - 2\alpha, \gamma_2)$ and $\tau_2 \in (\gamma_2, 2\alpha)$. Set $\beta_1 = \frac{\pi}{2} - \tau_1$ and $\beta_2 = \frac{\pi}{2} - (\pi - \tau_2) = \tau_2 - \frac{\pi}{2}$. With these choices of β_1 and β_2 , notice that

$$T(h^- \circ T_{\beta_1})(x_1, 0) \cdot (0, -1) = \cos(\tau_1) > \cos(\gamma_2), \quad \text{for } 0 < x_1 < 2 - r_0$$

and

$$T(h^+ \circ T_{\beta_2})(x_1, 0) \cdot (0, -1) = \cos(\tau_2) < \cos(\gamma_2), \quad \text{for } 0 < x_1 < 2 - r_0.$$

This implies that for $\delta_1 = \delta_1(\beta_1, \beta_2) > 0$ small enough,

$$T(h_{\beta_1}^-)(\mathbf{x}) \cdot \vec{\nu}(\mathbf{x}) > \cos(\gamma(\mathbf{x})) \quad \text{and} \quad T(h_{\beta_2}^+)(\mathbf{x}) \cdot \vec{\nu}(\mathbf{x}) < \cos(\gamma(\mathbf{x})) \quad (12)$$

for $\mathbf{x} \in \partial^- \Omega$ with $|\mathbf{x}| < \delta_1$, where $\vec{\nu}(\mathbf{x})$ is the exterior unit normal to Ω at $\mathbf{x} \in \partial \Omega$. (See Figure 5.) (We may also assume $\nu(\mathbf{x}) \cdot (1, 1) < 0$ for $\mathbf{x} \in \partial^+ \Omega$ with $|\mathbf{x}| < \delta_1$ and $\nu(\mathbf{x}) \cdot (1, -1) < 0$ for $\mathbf{x} \in \partial^- \Omega$ with $|\mathbf{x}| < \delta_1$ since $\alpha > \frac{\pi}{4}$.)

Let $\mu \in (0, \min\{\gamma_2 - (\pi - 2\alpha), 2\alpha - \gamma_2\})$ and set $\tau_1(\mu) = \pi - 2\alpha + \mu$ and $\tau_2(\mu) = 2\alpha - \mu$, so that $\beta_1 = \beta_2$. Let us write $\delta_1(\mu)$ for $\delta_1(\beta_1, \beta_2)$, h_μ^+ for $h_{\beta_2}^+$, h_μ^- for $h_{\beta_1}^-$ and Δ_μ for $\Delta_{\beta_1} = \Delta_{\beta_2}$. Since $\beta_1, \beta_2 \neq \pm \frac{\pi}{2}$, there exists $R = R(\mu) > 0$ such that $B(\mathcal{O}, R(\mu)) \cap \Omega^* \subset \Delta_\mu$ (where $B(\mathcal{O}, R) = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < R\}$).

Let us first assume that (A) holds and set $\mathbf{o} = \mathbf{o}_1 = \mathbf{o}_2$.

Claim: f is uniformly continuous on Ω_0 , where $\Omega_0 = \Omega^* \cap \Delta_\mu$.

Pf: For $r > 0$, set $B_r = \{\mathbf{u} \in \bar{E} : |\mathbf{u} - \mathbf{o}| < r\}$, $C_r = \{\mathbf{u} \in \bar{E} : |\mathbf{u} - \mathbf{o}| = r\}$ and let l_r be the length of the image curve $Y(C_r)$; also let $C'_r = G(C_r)$ and $B'_r = G(B_r)$. From the Courant-Lebesgue Lemma (e.g. Lemma 3.1 in [2]), we see that for each $\delta \in (0, 1)$, there exists a $\rho = \rho(\delta) \in (\delta, \sqrt{\delta})$ such that the arclength l_ρ of $Y(C_\rho)$

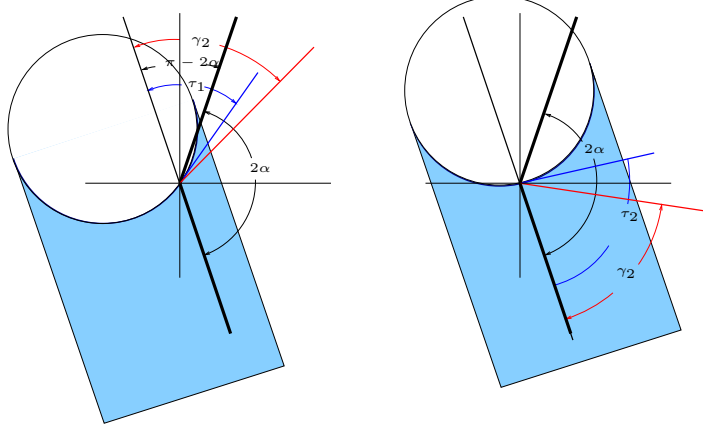


Figure 5: Left Δ_{β_1} : domain of $h_{\beta_1}^-$ Right Δ_{β_2} : domain of $h_{\beta_2}^+$

is less than $p(\delta)$. For $\delta > 0$, let $k(\delta) = \inf_{\mathbf{u} \in C_{\rho(\delta)}} c(\mathbf{u}) = \inf_{\mathbf{x} \in C'_{\rho(\delta)}} f(\mathbf{x})$ and $m(\delta) = \sup_{\mathbf{u} \in C_{\rho(\delta)}} c(\mathbf{u}) = \sup_{\mathbf{x} \in C'_{\rho(\delta)}} f(\mathbf{x})$; notice that $m(\delta) - k(\delta) \leq l_p < p(\delta)$.

For each $\delta \in (0, 1)$ with $\sqrt{\delta} < \min\{|\mathbf{o} - \mathbf{a}|, |\mathbf{o} - \mathbf{b}|\}$, there are two points in $C_{\rho(\delta)} \cap \partial E$; we denote these points as $\mathbf{e}_1(\delta) \in \mathbf{ob}$ and $\mathbf{e}_2(\delta) \in \mathbf{oa}$ and set $\mathbf{y}_1(\delta) = G(\mathbf{e}_1(\delta))$ and $\mathbf{y}_2(\delta) = G(\mathbf{e}_2(\delta))$. Notice that $C'_{\rho(\delta)}$ is a curve in $\bar{\Omega}$ which joins $\mathbf{y}_1 \in \partial^+ \Omega^*$ and $\mathbf{y}_2 \in \partial^- \Omega^*$ and $\partial\Omega \cap C'_{\rho(\delta)} \setminus \{\mathbf{y}_1, \mathbf{y}_2\} = \emptyset$; therefore there exists $\eta = \eta(\delta) > 0$ such that $B_{\eta(\delta)}(\mathcal{O}) = \{\mathbf{x} \in \Omega : |\mathbf{x}| < \eta(\delta)\} \subset B'_{\rho(\delta)}$ (see Figure 6).

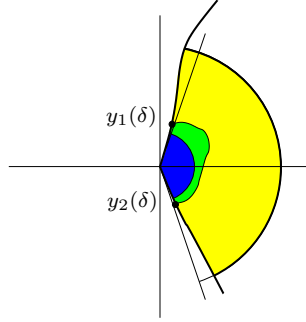


Figure 6: $B_{\eta(\delta)}(\mathcal{O})$ (blue region) and $B'_{\rho(\delta)}$ (blue & green regions)

Let $\epsilon > 0$. Choose $\delta > 0$ such that $\sqrt{\delta} < \min\{|\mathbf{o} - \mathbf{a}|, |\mathbf{o} - \mathbf{b}|\}$, $p(\delta) < \delta_1(\mu)$, $p(\delta) < R(\mu)$, and $p(\delta) + q(p(\delta)) < \frac{1}{2}\epsilon$. Pick a point $\mathbf{w} \in C'_{\rho(\delta)}$ and define $b_j^\pm : \Delta_\mu \rightarrow \mathbb{R}$ by

$$b^\pm(\mathbf{x}) = f(\mathbf{w}) \pm p(\delta) \pm h_\mu^\mp(\mathbf{x}), \quad \mathbf{x} \in \Delta_\mu.$$

From (10), (12) and the General Comparison Principle (e.g. [5], Theorem 5.1), we have

$$b^-(\mathbf{x}) < f(\mathbf{x}) < b^+(\mathbf{x}) \quad \text{for all } \mathbf{x} \in B'_{\rho(\delta)} \cap \Delta_\mu.$$

Thus if $\mathbf{x}_1, \mathbf{x}_2 \in \Omega_0$ satisfy $|\mathbf{x}_1| < \eta(\delta)$, $|\mathbf{x}_2| < \eta(\delta)$ and $|\mathbf{x}_1 - \mathbf{x}_2| < \eta(\delta)$, then

$$|f(\mathbf{x}_1) - f(\mathbf{x}_2)| < 2p(\delta) + 2q(p(\delta)) < \epsilon. \quad (13)$$

Since f is uniformly continuous on $\{\mathbf{x} \in \Omega^* : |\mathbf{x}| \geq \frac{1}{2}\eta(\delta)\}$, there exists a $\lambda > 0$ such that if $\mathbf{x}_1, \mathbf{x}_2 \in \Omega^*$ satisfy $|\mathbf{x}_1| \geq \frac{1}{2}\eta(\delta)$, $|\mathbf{x}_2| \geq \frac{1}{2}\eta(\delta)$ and $|\mathbf{x}_1 - \mathbf{x}_2| < \lambda$, then $|f(\mathbf{x}_1) - f(\mathbf{x}_2)| < \epsilon$. Now set $d = d(\epsilon) = \min\{\lambda, \frac{1}{2}\eta(\delta)\}$. If $\mathbf{x}_1, \mathbf{x}_2 \in \Omega_0$, $|\mathbf{x}_1 - \mathbf{x}_2| < d(\epsilon) \leq \frac{1}{2}\eta(\delta)$ and $|\mathbf{x}_1| < \frac{1}{2}\eta(\delta)$, then $|\mathbf{x}_1| < \eta(\delta)$ and $|\mathbf{x}_2| < \eta(\delta)$; hence $|f(\mathbf{x}_1) - f(\mathbf{x}_2)| < \epsilon$ by (13). Next, if $\mathbf{x}_1, \mathbf{x}_2 \in \Omega_0$, $|\mathbf{x}_1 - \mathbf{x}_2| < d(\epsilon) \leq \lambda$, $|\mathbf{x}_1| \geq \frac{1}{2}\eta(\delta)$ and $|\mathbf{x}_2| \geq \frac{1}{2}\eta(\delta)$, then $|f(\mathbf{x}_1) - f(\mathbf{x}_2)| < \epsilon$. Therefore, for all $\mathbf{x}_1, \mathbf{x}_2 \in \Omega_0$ with $|\mathbf{x}_1 - \mathbf{x}_2| < d(\epsilon)$, we have $|f(\mathbf{x}_1) - f(\mathbf{x}_2)| < \epsilon$. The claim is proven.

Notice that if $\theta(\mu) = \alpha - \mu$ ($= \tau_2(\mu) - \alpha = \pi - \alpha - \tau_1(\mu)$), then

$$\{(r \cos(\theta(\mu)), r \sin(\theta(\mu))) : r \geq 0\}$$

is the tangent ray to $\partial\Omega_0$ at \mathcal{O} and it follows from the Claim that $f \in C^0(\overline{\Omega_0})$; hence the radial limits $Rf(\theta)$ of f at \mathcal{O} exist for $\theta \in [-\alpha, \theta(\mu)]$ and the radial limits are identical (i.e. $Rf(\theta) = f(\mathcal{O})$ for all $\theta \in [-\alpha, \theta(\mu)]$, where $f(\mathcal{O})$ is the value at \mathcal{O} of the restriction of f to $\overline{\Omega_0}$). Since $\lim_{\mu \downarrow 0} \theta(\mu) = \alpha$, Theorem 1 is proven in this case.

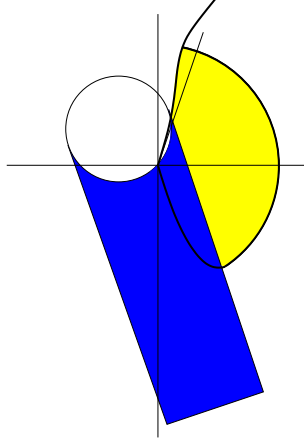


Figure 7: The domain (in blue) of the toroidal functions h_μ^\pm , $\alpha > \frac{\pi}{4}$.

Let us next assume that (B) holds. This part of the proof is essentially the same as the proof of case (B) in Theorem 1 of [4]. As in [4] and taking the hypothesis $\alpha \leq \frac{\pi}{2}$ into account, we see that

- (i) $c \in C^0(\overline{E} \setminus \{\mathbf{o}_1, \mathbf{o}_2\})$,
- (ii) there exist $\alpha_1, \alpha_2 \in [-\alpha, \alpha]$ with $\alpha_1 < \alpha_2$ such that $Rf(\theta)$ exists when $\theta \in (\alpha_1, \alpha_2)$, and
- (iii) Rf is strictly increasing or strictly decreasing on (α_1, α_2) .

Taking hypothesis (5) into account and using cylinders as in Case 3 of Step 1 in the proof of Theorem 1 of [7] (see Figure 2b in [8]) or using h_μ^\pm (see Figure 7), we see that in addition to (i)-(iii), we have

- (iv) $c \in C^0(\overline{E} \setminus \{\mathbf{o}_1\})$ and
- (v) $Rf(\theta)$ exists when $\theta \in [-\alpha, \alpha_2]$.

If $\alpha_2 = \alpha$, then Theorem 1 is proven. Otherwise, suppose $\alpha_2 < \alpha$ and fix $\delta_0 \in (0, \delta^*)$ and $\Omega_0 = \Omega^* \cap \Delta_\mu$ as before.

Claim: Suppose $\alpha_2 < \alpha$. Then f is uniformly continuous on Ω_0^+ , where

$$\Omega_0^+ \stackrel{\text{def}}{=} \{(r \cos(\theta), r \sin(\theta)) \in \Omega_0 : 0 < r < \delta^*, \alpha_2 < \theta < \pi\}.$$

Notice that the restriction of Y to $G^{-1}(\overline{\Omega_0^+})$ maps only one point, \mathbf{o}_1 , to $\mathcal{O} \times \mathbb{R}$ and so the proof of this claim is the same as the proof of the previous Claim. Thus $f \in C^0(\overline{\Omega_0^+})$; since $\lim_{\mu \downarrow 0} \theta(\mu) = \alpha$, we see that

$$Rf(\theta) = \lim_{\tau \uparrow \alpha_2} Rf(\tau) \quad \text{for all } \theta \in [\alpha_2, \alpha).$$

Thus Theorem 1 is proven.

4 Proof of Theorem 2

Suppose (6) does not hold. Since $\pi - 2\alpha - \lambda_1 < \gamma_2 < \pi + 2\alpha - \lambda_2$, we can choose $\tau_1, \tau_2 \in (0, \pi)$ such that $\tau_1 \in (\pi - 2\alpha - \lambda_1, \gamma_2)$ and $\tau_2 \in (\gamma_2, \pi + 2\alpha - \lambda_2)$. Set $\beta_1 = \frac{\pi}{2} - \tau_1$ and $\beta_2 = \tau_2 - \frac{\pi}{2}$. (See Figures 8 and 9.) With these choices of β_1 and β_2 , notice that

$$T(h^- \circ T_{\beta_1})(x_1, 0) \cdot (0, -1) = \cos(\tau_1) > \cos(\gamma_2), \quad \text{for } 0 < x_1 < 2 - r_0$$

and

$$T(h^+ \circ T_{\beta_2})(x_1, 0) \cdot (0, -1) = \cos(\tau_2) < \cos(\gamma_2), \quad \text{for } 0 < x_1 < 2 - r_0.$$

This implies that for $\delta_1 = \delta_1(\beta_1, \beta_2) > 0$ small enough,

$$T(h_{\beta_1}^-)(\mathbf{x}) \cdot \vec{\nu}(\mathbf{x}) > \cos(\gamma(\mathbf{x})) \quad \text{and} \quad T(h_{\beta_2}^+)(\mathbf{x}) \cdot \vec{\nu}(\mathbf{x}) < \cos(\gamma(\mathbf{x})) \quad (14)$$

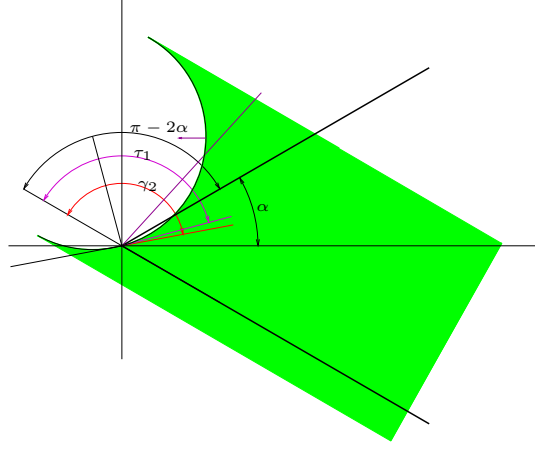


Figure 8: $\alpha = \frac{\pi}{6}$, $\lambda_1 = 0$, $\lambda_2 = \frac{\pi}{2}$, $\gamma_2 = \frac{7\pi}{9}$, and $\tau_1 = \frac{27\pi}{36}$. The domain of $h_{\beta_1}^-$ is the green region.

for $\mathbf{x} \in \partial^-\Omega$ with $|\mathbf{x}| < \delta_1$, where $\vec{\nu}(\mathbf{x})$ is the exterior unit normal to Ω at $\mathbf{x} \in \partial\Omega$. (See Figures 5, 8 and 9.)

Notice that the tangent plane at $(0, 0, 0)$ to the surface $\{(\mathbf{x}, h_{\beta_1}^-(\mathbf{x})) : \mathbf{x} \in \Delta_{\beta_1}\}$ is a vertical plane with (downward oriented) unit normal $\vec{n} = (-\sin(\tau_1 + \alpha), -\cos(\tau_1 + \alpha), 0)$ and $\lim_{\partial^+\Omega \ni \mathbf{x} \rightarrow \mathcal{O}} \vec{\nu}(\mathbf{x}) = (-\sin(\alpha), \cos(\alpha), 0)$. Suppose $\tau_1 + 2\alpha \leq \pi$. Then

$$\lim_{\partial^+\Omega \ni \mathbf{x} \rightarrow \mathcal{O}} \vec{n} \cdot \vec{\nu}(\mathbf{x}) = -\cos(\tau_1 + 2\alpha) > -\cos(\pi - \lambda_1) = \cos(\lambda_1)$$

since $\tau_1 + 2\alpha > \pi - \lambda_1$; since $\liminf_{\partial^+\Omega \ni \mathbf{x} \rightarrow \mathcal{O}} \gamma(\mathbf{x}) \geq \lambda_1$, this implies that for some $\delta_2 > 0$ small enough,

$$T(h_{\beta_1}^-)(\mathbf{x}) \cdot \vec{\nu}(\mathbf{x}) > \cos(\gamma(\mathbf{x})) \quad \text{for } \mathbf{x} \in \partial^+\Omega \text{ with } |\mathbf{x}| < \delta_2. \quad (15)$$

If $\tau_1 + 2\alpha > \pi$, then λ_1 doesn't matter and we argue as in the proof of Theorem 1; see Figure 8 for an illustration of this case.

Now the tangent plane at $(0, 0, 0)$ to the surface $\{(\mathbf{x}, h_{\beta_2}^+(\mathbf{x})) : \mathbf{x} \in \Delta_{\beta_2}\}$ is a vertical plane with (downward oriented) unit normal $\vec{m} = (\sin(\tau_2 - \alpha), -\cos(\tau_2 - \alpha), 0)$ and $\lim_{\partial^+\Omega \ni \mathbf{x} \rightarrow \mathcal{O}} \vec{\nu}(\mathbf{x}) = (-\sin(\alpha), \cos(\alpha), 0)$. Suppose $\tau_2 \geq 2\alpha$. Then

$$\lim_{\partial^+\Omega \ni \mathbf{x} \rightarrow \mathcal{O}} \vec{m} \cdot \vec{\nu}(\mathbf{x}) = -\cos(\tau_2 - 2\alpha) < -\cos(\pi - \lambda_2) = \cos(\lambda_2)$$

since $\tau_2 - 2\alpha < \pi - \lambda_2$; since $\limsup_{\partial^+\Omega \ni \mathbf{x} \rightarrow \mathcal{O}} \gamma(\mathbf{x}) \leq \lambda_2$, this implies that for some $\delta_3 > 0$ small enough,

$$T(h_{\beta_1}^+)(\mathbf{x}) \cdot \vec{\nu}(\mathbf{x}) < \cos(\gamma(\mathbf{x})) \quad \text{for } \mathbf{x} \in \partial^+\Omega \text{ with } |\mathbf{x}| < \delta_3. \quad (16)$$

If $\tau_2 < 2\alpha$, then λ_2 doesn't matter and we argue as in the proof of Theorem 1.

Now set $\delta_4 = \min\{\delta_1, \delta_2, \delta_3\}$. The proof of Theorem 2 now follows essentially as in the proof of Theorem 1.

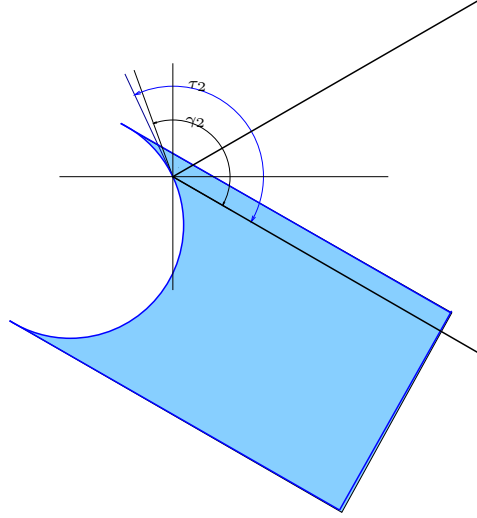


Figure 9: $\alpha = \frac{\pi}{6}$, $\lambda_1 = 0$, $\lambda_2 = \frac{\pi}{2}$, $\gamma_2 = \frac{7\pi}{9}$, and $\tau_2 = \frac{29\pi}{36}$. The domain of $h_{\beta_2}^+$ is the blue region.

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